1. Let $p = (x, y, z)$, $F(p) = |p|^2 - \langle p, a \rangle^2$

Then $S = F^{-1}(r)$

$\nabla F|_p = 2p - 2\langle p, a \rangle a$

$v \in T_p S$ if and only if $v \cdot \nabla F = 0$

(i.e. $2\langle p, v \rangle - 2\langle p, a \rangle \langle a, v \rangle = 0$)

So $T_p S = \{ v \in \mathbb{R}^3 : \langle p, v \rangle - \langle p, a \rangle \langle a, v \rangle = 0 \}$

Consider the vector $p - \langle p, a \rangle a$, $p \in S$

Then for any $v \in T_p S$, $\langle p - \langle p, a \rangle a, v \rangle = 0$

$p - \langle p, a \rangle a \parallel N(p)$

$$
\begin{aligned}
|p - \langle p, a \rangle a|^2 &= \langle p - \langle p, a \rangle a, p - \langle p, a \rangle a \rangle \\
&= |p|^2 - 2\langle p, a \rangle \langle p, a \rangle + \langle p, a \rangle^2 \langle a, a \rangle \qquad (|a| = 1) \\
&= |p|^2 - \langle p, a \rangle^2 \\
&= r^2
\end{aligned}
$$

So $p - \langle p, a \rangle a = \pm r N(p)$

Hence all the normal lines of S cut the axis orthogonally.

For the converse,

Suppose all normal lines of S intersect a fixed straight line $l \subset \mathbb{R}^3$ orthogonally.

We assume l pass through the origin,

Let $p \in S$, $\exists$ smooth $X : U \subseteq \mathbb{R}^2 \to V$, with V is nbd of p in S

Let $\vec{a}$ be unit vector $\parallel l$

Then $X(u,v) - r(u,v) N(u,v) = \langle X(u,v), a \rangle a$

$X_u - r_u N - r N_u = \langle X_u, a \rangle a$

$$
\begin{aligned}
0 = \langle X_u, a \rangle \langle a, N \rangle &= \langle X_u, N \rangle - r_u \langle N, N \rangle - r \langle N_u, N \rangle \\
&= -r_u
\end{aligned}
$$

$r_u = 0$

Similarly , $r_v = 0$

So r is constant on V

Since S is connected , r is constant on S

$p = \langle p, a\rangle a + r N(p)$

$|p|^2 = \langle p, a\rangle^2 + r^2$

$|p|^2 - \langle p, a\rangle^2 = r^2 \quad \forall\, p \in S$

2. $\phi : S_1 \to S_2 \quad , \quad \phi(x,y,z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$

3. Let $F_1(x,y,z) = x^2+y^2+z^2 - ax$

$F_2(x,y,z) = x^2+y^2+z^2 - by$

$F_3(x,y,z) = x^2+y^2+z^2 - cz$

Then $\nabla F_1 = (2x-a; 2y, 2z)$, $\nabla F_2 = (2x, 2y-b; 2z)$; $\nabla F_3 = (2x, 2y, 2z-c)$

$S_1 = F_1^{-1}(0)$ and 0 is a regular value of F_1 , so S_1 is surface

Similar for showing S_2, S_3 are surface

$\nabla F_1 \cdot F_2 = 4x^2 - 2ax + 4y^2 - 2by + 4z^2$

$= 2(x^2+y^2+z^2-ax) + 2(x^2+y^2+z^2-by)$

$= 0$ at the intersection points of S_1 and S_2

Similar for $\nabla F_1 \cdot \nabla F_3$, $\nabla F_2 \cdot \nabla F_3$

4. S lies on one side of P, i.e. $\langle p, n\rangle \geq c$ for some unit normal n to P
some constant c.
$\forall\, p \in S$

$\langle p, n\rangle = c$ if and only if $p \in P$

Let $q \in P \cap S$, $v \in T_q S$

$\exists \ \alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = q$, $\alpha'(0) = v$

Then $\langle \alpha(t), n\rangle \geq c \quad \langle \alpha(0), n\rangle = c$

$\langle \alpha(t), n\rangle$ has local minimum at $t = 0$

$\langle \alpha'(0), n\rangle = 0$

$\langle v, n\rangle = 0$, hence n is normal to $T_q S$

Therefore $T_q P = T_q S$

5. Since S is compact surface

$\exists\ p, q \in S$ such that $|p-q|^2 = \max_{x,y \in S} |x-y|^2$

We want $p-q \perp T_pS$ and $p-q \perp T_qS$

the straight line from p to q is the required straight line ℓ

$\vdash$: $p-q \perp T_pS$

Let $v \in T_pS$. , $\exists\ \alpha : (-\varepsilon, \varepsilon) \to S$ such that $\alpha(0) = p$, $\alpha'(0) = v$

Consider $\langle \alpha(t) - q , \alpha(t) - q \rangle$ which has local maximum at $t = 0$

$\langle \alpha'(0), \alpha(0) - q \rangle = 0$

$\langle v, p - q \rangle = 0$

Hence $p-q \perp T_pS$

Same argument to $p-q \perp T_qS$